

On algebraic integers in short intervals and near smooth curves

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Abstract

In 1970 A. Baker and W. Schmidt introduced regular systems of numbers and vectors, showing that the set of real algebraic numbers forms a regular system on any fixed interval. This fact was used to prove several important results in the metric theory of transcendental numbers. In this paper the concept of a regular system is applied to the set of algebraic integers α in short intervals of length depending on the height of α .

1 Introduction

Many problems in the theory of Diophantine approximation are related to the distribution of algebraic numbers and algebraic integers [14, 24]. In this paper we study the distribution of algebraic integers on the real line and

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the distribution of the points with conjugate algebraic integer coordinates in the Euclidean plane.

Let $P(t) = a_n t^n + \dots + a_1 t + a_0$, $a_i \in \mathbb{Z}$ be a polynomial with integer coefficients of degree $\deg P = n$. Given a polynomial P , let $H(P) = \max_{0 \leq j \leq n} |a_j|$ denote the height of P .

Let us consider an irreducible polynomial $P \in \mathbb{Z}[t]$ with coprime coefficients: $\gcd(|a_n|, \dots, |a_0|) = 1$. The roots of this polynomial are algebraic numbers α of degree n and height $H(\alpha) = H(P)$. When $a_n = 1$, roots of an irreducible polynomial $P(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0$ are called algebraic integers α of degree n and height $H(\alpha) = H(P)$. Let $\#S$ denote the cardinality of a finite set S and $\mu_k D$ be the Lebesgue measure of a measurable set $D \subset \mathbb{R}^k$, $k \in \mathbb{N}$. We define the following class of polynomials:

$$\mathcal{P}_n(Q) = \{P \in \mathbb{Z}[t] : \deg P \leq n, H(P) \leq Q\},$$

where $Q > Q_0(n)$ is a sufficiently large integer. The notation $c_j > 0$, $j \in \mathbb{N}$ is going to be used for certain positive real numbers which depend on n but don't depend on $H(P)$ or Q .

Over the last 20 years, new insights about the distribution of algebraic numbers have been made. In particular, lower and upper bounds for the distances between algebraically conjugate numbers and the roots of different integer polynomials were proposed in the papers [4, 13, 19, 10].

Let us consider an interval $I \subset [-\frac{1}{2}; \frac{1}{2}]$ of length $|I| = c_1 Q^{-1}$. It would be interesting to know whether an interval I of such length contains algebraic numbers α of degree $\deg \alpha \leq n$ and height $H(\alpha) \leq Q$. For $n = 3$, this question was answered in the paper by V. Bernik, N. Budarina and H. O'Donnell [9] from 2012, and a generalization for an arbitrary n was proved by V. Bernik and F. Götze in 2015 [6]. Another result in the paper [6] states that for any integer $Q \geq 1$ there exists an interval I of length $|I| = \frac{1}{2} Q^{-1}$, which doesn't contain any algebraic numbers α of an arbitrary degree and height $H(\alpha) \leq Q$. On the other hand, for a sufficiently large height $Q > Q_0(n)$ and a sufficiently large constant c_1 , any interval I of length $|I| \geq c_1 Q^{-1}$ contains at least $c_2 Q^{n+1} |I|$ real algebraic numbers α of degree $\deg \alpha \leq n$ and height $H(\alpha) \leq Q$. Furthermore, a regular system can be constructed from these algebraic numbers [12].

In this paper we are going to obtain a similar result for algebraic integers.

Theorem 1.1. *For any integer $Q \geq 1$ there exists an interval I of length $|I| = \frac{1}{2} \cdot Q^{-1}$ which doesn't contain algebraic integers α of height $H(\alpha) \leq Q$ and arbitrary degree $n \geq 2$.*

It is easy to see that Theorem 1.1 follows from the results presented in [6], since algebraic integers form a subset of the set of algebraic numbers.

Theorem 1.2. *Let the constant c_3 and the number $Q > Q_0(n)$ be sufficiently large. Then there exists a constant c_4 such that any interval I of length $|I| \geq c_3 Q^{-1}$ contains at least $c_4 Q^n |I|$ real algebraic integers α of degree $\deg \alpha = n$, $n \geq 2$ and height $H(\alpha) \leq Q$.*

We are also going to show that the set of real algebraic integers of degree n forms a regular system in short intervals.

Definition. Let Γ be a countable set of real numbers and $N : \Gamma \rightarrow \mathbb{R}$ be a positive function. The pair (Γ, N) is called a *regular system* if there exists a constant $c_5 = c_5(\Gamma, N) > 0$ such that for any interval $I \subset \mathbb{R}$ the following property is satisfied: for a sufficiently large number $T_0 = T_0(\Gamma, N, I) > 0$ and an arbitrary integer $T > T_0$ there exist $\gamma_1, \gamma_2, \dots, \gamma_t \in \Gamma \cap I$ such that

- 1) $N(\gamma_i) \leq T, \quad 1 \leq i \leq t,$
- 2) $|\gamma_i - \gamma_j| > T^{-1}, \quad 1 \leq i < j \leq t,$
- 3) $t > c_5 T |I|.$

Obviously, the set of rational numbers p/q together with the function $N(p/q) := q^2$ is a regular system. Similarly, the set of real algebraic numbers α of degree n forms a regular system with respect to the function $N(\alpha) = H(\alpha)^{n+1}$ (see, e.g., [2, 1]).

Regular systems of algebraic numbers are used to obtain lower bounds for the Hausdorff dimension of various algebraic number sets, such as the set of real algebraic numbers with a given measure of transcendence [1] and the set of real numbers lying close to zeros of integer combinations of non-degenerate functions of given order [17]. Another application of regular systems is the proof of Khinchine-type theorems in the case of divergence [2, 5, 8].

A similar result related to the distribution of points with algebraically conjugate coordinates in the Euclidean plane was obtained by V. Bernik, F. Götze and O. Kukso in the paper [7]. Let us consider a rectangle $E = I_1 \times I_2 \subset [-\frac{1}{2}; \frac{1}{2}]^2$, where I_1, I_2 are intervals of lengths $|I_1| = Q^{-\kappa_1}$, $|I_2| = Q^{-\kappa_2}$ and $0 < \kappa_1, \kappa_2 < \frac{1}{2}$. Furthermore, we demand that

$$E \cap \{(x, y) \in \mathbb{R}^2 : |x - y| \leq \varepsilon\} = \emptyset,$$

where $\varepsilon > 0$ is a sufficiently small constant. This choice of the rectangle E will simplify our argument. A point (α, β) is called an *algebraic point* if α

and β are algebraically conjugate numbers, and an *algebraic integer point* if α and β are algebraically conjugate integers. In the paper [7] it is shown that for $Q > Q_0(n)$ any rectangle E of size $\mu_2 E = |I_1| \cdot |I_2| = Q^{-\kappa_1 - \kappa_2}$, $0 < \kappa_1, \kappa_2 < \frac{1}{2}$ contains at least $c_6 Q^{n+1} \mu_2 E$ algebraic points (α, β) of degree $\deg \alpha = \deg \beta \leq n$, $n \geq 2$ and height $H(\alpha) = H(\beta) \leq Q$.

Let us prove that the same estimate holds for algebraic integer points.

Theorem 1.3. *For a sufficiently large $Q > Q_0(n)$, there exists a constant c_7 such that any rectangle $E = I_1 \times I_2$ of size $\mu_2 E = |I_1| \cdot |I_2| = Q^{-\kappa_1 - \kappa_2}$, $0 < \kappa_1, \kappa_2 < \frac{1}{2}$, contains at least $c_7 Q^n \mu_2 E$ algebraic integer points (α, β) of degree $\deg \alpha = \deg \beta = n$, $n \geq 4$ and height $H(\alpha) = H(\beta) \leq Q$.*

A number of interesting problems arise when distribution of algebraic points close to smooth curves is investigated [21]. Let us mention several recent results in this area. Upper and lower bounds on the number of rational points near smooth curves of the same order have been obtained in the papers [3] and [25]. The paper [7] from 2014 presents lower estimates for the number of algebraic points of arbitrary degree in neighborhoods of smooth curves.

Our main result is a lower estimate for the number of algebraic integer points of arbitrary degree close to smooth curves, thus specializing the results of the paper [7] for algebraic integers.

Theorem 1.4. *Let $y = f(x)$ be a continuous differentiable function in an interval $J = [a, b]$ and let*

$$L_J(Q, \lambda) = \{(x, y) \in \mathbb{R}^2 : x_1 \in J, |y - f(x)| < Q^{-\lambda}\},$$

where $0 < \lambda < \frac{1}{2}$. Then for $Q > Q_0(n, J, f, \lambda)$ there are exist $c_8(n, J, f) Q^{n-\lambda}$, $c_8(n, J, f) > 0$ algebraic integer points (α, β) of degree $\deg \alpha = \deg \beta = n$, $n \geq 4$ and of height $H(\alpha) = H(\beta) \leq Q$ such that $(\alpha, \beta) \in L_J(Q, \lambda)$.

Proof. Consider a graph of the function $y = f(x)$ and the strip $L_J(Q, \lambda)$ for a fixed λ , $0 < \lambda < \frac{1}{2}$. Divide the interval J into sub-intervals $J_i = [x_{i-1}, x_i]$ of length $|J_i| = Q^{-\lambda}$, $i = \overline{1, m}$, where $m > (a - b)Q^\lambda - 1 > c_9(J)Q^\lambda$ for $Q > Q_0$. Let $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$ denote their midpoints. Consider rectangles

$$E_i = \{(x, y) \in \mathbb{R}^2 : |x - \bar{x}_i| \leq c_{10}(n, f)Q^{-\lambda}; |y - f(\bar{x}_i)| \leq c_{11}(n, f)Q^{-\lambda}\},$$

where $c_{10}(n, f)$ and $c_{11}(n, f)$ are chosen so that the rectangles E_i are fully enclosed in

$$T_i = \left\{ (x, y) \in \mathbb{R}^2 : |x - \bar{x}_i| \leq \frac{1}{2}Q^{-\lambda}; |y - f(x)| \leq Q^{-\lambda} \right\},$$

where $i = \overline{1, m}$.

It follows from Theorem 3 that every rectangle E_i , $i = \overline{1, m}$ contains at least $c_{12}(n, f)Q^{n-2\lambda}$ algebraic integer points of degree n and height at most Q . Therefore, since $m > c_9(J)Q^\lambda$, there exist at least $c_8(n, J, f)Q^{n-\lambda}$ algebraic integer points $(\alpha, \beta) \in L_J(Q, \lambda)$. \square

2 Auxiliary statements

This section contains several lemmas which will be used to prove Theorems 1.2 and 1.3. Some of them are related to geometry of numbers, see [15]. The first paper discussing approximation by algebraic integers was written by H. Davenport and W.M Schmidt [16]. Recently, their approach has been reinterpreted by Y. Bugeaud [11]. In our paper we are going to apply some of the ideas in Y. Bugeaud's paper.

Lemma 2.1 (Minkowski's 2nd theorem on successive minima). *Let K be a bounded central symmetric convex body in \mathbb{R}^n with successive minima τ_1, \dots, τ_n . Then*

$$\frac{2^n}{n!} \leq \tau_1 \tau_2 \dots \tau_n V(K) \leq 2^n.$$

For a proof, see [15, pp. 203], [20, pp. 59].

Lemma 2.2 (Bertrand postulate). *For any integer $n \geq 2$ there exists a prime p such that $n < p < 2n$.*

Proved by P. Chebyshev in 1850. A proof can be found, for example, in [22, Theorem 2.4].

Lemma 2.3 (Eisenstein criterion). *Let P denote a polynomial with integer coefficients,*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

If there exists a prime number p such that:

$$(2.1) \quad \begin{cases} a_n \not\equiv 0 \pmod{p}, \\ a_i \equiv 0 \pmod{p}, & i = 0, \dots, n-1 \\ a_0 \not\equiv 0 \pmod{p^2}, \end{cases}$$

then P is irreducible over the rational numbers.

For a proof see [18].

Lemma 2.4. *Consider a point $x \in \mathbb{R}$ and a polynomial P with zeros $\alpha_1, \alpha_2, \dots, \alpha_n$, where $|x - \alpha_1| = \min_i |x - \alpha_i|$. Then*

$$|x - \alpha_1| \leq n|P(x)| \cdot |P'(x)|^{-1}.$$

Proof. Consider the polynomial P and its derivative P' at the point x . Since

$$|P'(x)||P(x)|^{-1} \leq \sum_{i=1}^n |x - \alpha_i|^{-1} \leq n|x - \alpha_1|^{-1},$$

we have

$$|x - \alpha_1| \leq n|P(x)| \cdot |P'(x)|^{-1}.$$

□

Lemma 2.5 (see [6]). *Let $\mathcal{L}_n = \mathcal{L}_n(Q, \delta_0, I)$ be the set of points $x \in I$ such that a system of equations*

$$\begin{cases} |P(x)| < Q^{-n}, \\ |P'(x)| < \delta_0 Q, \end{cases}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$. For any sufficiently small constant $\delta_0 = \delta_0(n) > 0$, there exists a sufficiently large constant c_{13} such that

$$\mu_1 \mathcal{L}_n < \frac{1}{4} \cdot |I|$$

for all intervals I satisfying the condition $|I| > c_{13} Q^{-1}$.

In [6] it is shown that it suffices to take $\delta_0(n) = 2^{-n-8}n^{-2}$.

Lemma 2.6 (see [7]). *Given positive v_1 and v_2 subject to the condition $v_1 + v_2 = n - 1$, let $\mathcal{M}_n = \mathcal{M}_n(Q, \delta_0, E, v_1, v_2)$ be set of points $(x, y) \in E$, such that a system of equations*

$$(2.2) \quad \begin{cases} |P(x)| < Q^{-v_1}, & |P(y)| < Q^{-v_2}, \\ \min_i \{|P'(x)|, |P'(y)|\} < \delta_0 Q, \end{cases}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$. Then for $\delta_0 = \delta_0(n) < 2^{-n-40}n^{-4}$ we have

$$\mu_2 \mathcal{M}_n < \frac{1}{4} \cdot \mu_2 E$$

for all rectangles $E = I_1 \times I_2$ such that $|I_i| = Q^{-\kappa_i}$, $0 < \kappa_i < \frac{1}{2}$, $i = 1, 2$.

Remark 1. For every point $(x, y) \in E \subset [-\frac{1}{2}, \frac{1}{2}]$ and for all polynomials $P \in \mathcal{P}_n(Q)$ we have the following estimate:

$$|P(x)| < 2Q, \quad |P(y)| < 2Q.$$

Hence the values v_1 and v_2 lie between -1 and n .

Remark 2. It is easy to see (for example from Lemma 2.4) that for a fixed polynomial P the set of points $(x, y) \in \mathbb{R}^2$ satisfying the system (2.2) is contained in a rectangle $\sigma_P = J_1 \times J_2$ of measure smaller than $\frac{1}{4} \cdot \mu_2 E$, see [7]. In the case where $I_1 \subset J_1$ or $I_2 \subset J_2$, we consider the rectangle $I_1 \times J_2$ or $J_1 \times I_2$ instead of the rectangle σ_P when estimating the measure of \mathcal{M}_n .

3 Proof of Theorem 1.2

Let $\mathcal{L}_{n-1} = \mathcal{L}_{n-1}(Q, \delta_0, I)$ be the set of $x \in I$ such that the system

$$(3.1) \quad \begin{cases} |P(x)| < Q^{-n+1}, \\ |P'(x)| < \delta_0 Q, \end{cases}$$

has a solution in polynomials $P \in \mathcal{P}_{n-1}(Q)$. From Lemma 2.5 it follows that the measure of the set \mathcal{L}_{n-1} can be estimated as

$$\mu \mathcal{L}_{n-1} \leq \frac{1}{4} \cdot |I|,$$

where $|I| > c_{13} Q^{-1}$ for $Q > Q_0(n)$ and $\delta_0 < 2^{-n-7}(n-1)^{-2}$.

Let us consider the set $B_1 = I \setminus \mathcal{L}_{n-1}$. Since the first inequality of the system (3.1) has solution in polynomials $P \in \mathcal{P}_{n-1}(Q)$ for any $x \in I$, we can say that for any $x_0 \in B_1$ and any polynomial $P \in \mathcal{P}_{n-1}(Q)$, the system of inequalities

$$\begin{cases} |P(x_0)| < Q^{-n+1}, \\ |P'(x_0)| \geq \delta_0 Q, \end{cases}$$

holds, and $\mu_1 B_1 \geq \frac{3}{4} \cdot |I|$.

Consider an arbitrary point $x_0 \in B_1$ and examine successive minima τ_1, \dots, τ_n of the compact convex set defined by inequalities

$$(3.2) \quad \begin{cases} |a_{n-1}x_0^{n-1} + \dots + a_1x_0 + a_0| \leq Q^{-n+1}, \\ |(n-1)a_{n-1}x_0^{n-2} + \dots + 2a_2x_0 + a_1| \leq Q, \\ |a_{n-1}|, \dots, |a_2| \leq Q. \end{cases}$$

Let $\tau_1 \leq \delta_0$. Then there exists a polynomial $P_0 \in \mathcal{P}_{n-1}(Q)$ such that inequalities

$$\begin{cases} |P_0(x_0)| \leq \delta_0 Q^{-n+1} < Q^{-n+1}, \\ |P'_0(x_0)| \leq \delta_0 Q, \\ H(P_0) \leq \delta_0 Q < Q, \end{cases}$$

are satisfied. This leads to a contradiction since $x_0 \notin \mathcal{L}_{n-1}$, following us to conclude that $\tau_1 > \delta_0$. Since the volume of the compact convex set defined

by the inequalities (3.2) is at least 2^n , it follows from Lemma 2.1 that $\tau_1 \dots \tau_n \leq 1$ and $\tau_{n-1} \leq \delta_0^{-n+1}$. Thus we can choose n linearly independent polynomials $P_i \in \mathcal{P}_{n-1}(Q)$, $i = \overline{1, n}$, such that the system of inequalities

$$(3.3) \quad \begin{cases} |P_i(x_0)| \leq \delta_0^{-n+1} Q^{-n+1}, \\ |P'_i(x_0)| \leq \delta_0^{-n+1} Q, \\ H(P_i) \leq \delta_0^{-n+1} Q. \end{cases}$$

holds. Using well-known estimates from geometry of numbers (see [15, pp. 219]) for the polynomials $P_i(t) = a_{i,n-1}t^{n-1} + \dots + a_{i,1}t + a_{i,0}$, $i = \overline{1, n}$ we obtain the inequality

$$\Delta = \det |(a_{i,j-1})_{i,j=1}^n| \leq n!.$$

It follows from Lemma 2.2 that there exists a prime p which doesn't divide Δ such that

$$(3.4) \quad n! < p < 2n!.$$

Consider the following system of linear equations in n variables $\theta_1, \dots, \theta_n$:

$$(3.5) \quad \begin{cases} x_0^n + p \cdot \sum_{i=1}^n \theta_i P_i(x_0) = p(n+1) \cdot \delta_0^{-n+1} Q^{-n+1}, \\ nx_0^{n-1} + p \cdot \sum_{i=1}^n \theta_i P'_i(x_0) = pQ + p \cdot \sum_{i=1}^n |P'_i(x_0)|, \\ \sum_{i=1}^n \theta_i a_{i,j} = 0, \quad 2 \leq j \leq n-1. \end{cases}$$

In order to find the determinant of this system, let us transform it as follows. Multiply the equations numbered as $k = 3, \dots, n$ by $p \cdot x_0^{k-1}$ and subtract them from the first equation of the system (3.5). Similarly, multiply the equations numbered as $k = 3, \dots, n$ by $p \cdot (k-1)x_0^{k-2}$ and subtract them from the second equation. After these transformations the determinant of the system (3.5) can be written as

$$\hat{\Delta} = p^2 \cdot \begin{vmatrix} a_{1,1}x_0 + a_{1,0} & \dots & a_{n,1}x_0 + a_{n,0} \\ a_{1,1} & \dots & a_{n,1} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix}$$

Since the polynomials P_i are linearly independent, and the determinant of the system (3.5) is equal to $p^2\Delta \neq 0$, this system of equations has a unique solution $(\theta_1, \dots, \theta_n)$. Let us choose n integers t_1, \dots, t_n such that

$$(3.6) \quad |\theta_i - t_i| \leq 1, \quad (i = 1, \dots, n).$$

Consider the following polynomial of degree n with integer coefficients:

$$P(t) = t^n + p \cdot \sum_{i=1}^n t_i P_i(t) = t^n + p \cdot (a_{n-1}t^{n-1} + \dots + a_1t + a_0),$$

where $a_j = \sum_{i=1}^n t_i a_{i,j}$, $0 \leq j \leq n-1$.

The polynomial P is irreducible if it satisfies the conditions of Lemma 2.3. Let us show that there exists a respective combinations of the coefficients t_j . Clearly, the first and the second conditions of (2.1) hold for any t_j . It remains to show that $a_0 = t_1 a_{1,0} + \dots + t_n a_{n,0}$ isn't divisible by p . Since p doesn't divide Δ , there exists a number $1 \leq i \leq n$ such that $a_{i,0}$ is not divisible by p . From the condition (3.6), we have two possible values for t_i , which can be denoted as t_i^1 , $t_i^2 = t_i^1 + 1$. Since $a_{i,0}$ isn't divisible by p , either $a_0^1 = t_1 a_{1,0} + \dots + a_{i,0} t_i^1 + \dots + a_{n,0} t_n$ or $a_0^2 = t_1 a_{1,0} + \dots + a_{i,0} t_i^2 + \dots + a_{n,0} t_n$ is also not divisible by p . Therefore, choosing t_i in this manner yields an irreducible polynomial P .

In the following we shall estimate $|P(x_0)|$, $|P'(x_0)|$ and $H(P)$.

From the inequalities (3.3), (3.6) and the first equation of the system (3.5) it follows that

$$(3.7) \quad p\delta_0^{-n+1}Q^{-n+1} \leq |P(x_0)| \leq p(2n+1)\delta_0^{-n+1}Q^{-n+1}.$$

Similarly, by inequalities (3.3), (3.6) and the second equation of the system (3.5) we have

$$(3.8) \quad pQ \leq |P'(x_0)| \leq (p + 2pn\delta_0^{-n+1})Q.$$

In view of inequalities (3.3), (3.6) and the remaining equations of the system (3.5), we have

$$(3.9) \quad |a_j| \leq n\delta_0^{-n+1}Q, \quad 2 \leq j \leq n-1.$$

Finally, we need to estimate $|a_0|$ and $|a_1|$. Here we use the estimates (3.7)—(3.9) and the inequality $|x_0| \leq \frac{1}{2}$. This yields the estimates

$$(3.10) \quad |a_1| \leq |P'(x_0)| + \sum_{j=2}^n |a_j| \leq (p + (2pn + n^2)\delta_0^{-n+1})Q,$$

$$(3.11) \quad |a_0| \leq |P(x_0)| + |a_1| + \sum_{j=2}^n |a_j| \leq (p + (p(4n+1) + n^2)\delta_0^{-n+1})Q.$$

From the estimates (3.9)—(3.11) and the inequality (3.4) we conclude that

$$(3.12) \quad H(P) \leq 6(n+1)!\delta_0^{-n+1}Q.$$

Let us consider the roots $\alpha_1, \dots, \alpha_n$ of the polynomial P , where $|x_0 - \alpha_1| = \min_i |x_0 - \alpha_i|$. In view of Lemma 2.4, the following estimate holds

$$|x_0 - \alpha_1| \leq n|P(x_0)||P'(x_0)|^{-1}.$$

By inequalities (3.7) and (3.8) we have

$$(3.13) \quad |x_0 - \alpha_1| \leq n(2n+1)\delta_0^{-n+1}Q^{-n} = c_{14}Q^{-n},$$

where $c_{14} = n(2n+1)\delta_0^{-n+1}$.

If α_1 is a complex root, then its conjugate is also a root of P . Hence, by (3.12), (3.13) and well-known estimates for the roots of the polynomial P , $|\alpha_i| \leq H(P) + 1$, $1 \leq i \leq n$ (see [23, pp. 2, Theorem 1.1.2]), we obtain that

$$|P(x_0)| = \prod_{i=1}^n |x_0 - \alpha_i| \leq c_{14}^2 Q^{-2n} \cdot (2 + 6(n+1)!\delta_0^{-n+1}Q)^{n-2}.$$

This inequality contradicts (3.7) for $Q > Q_0(n)$. Thus, α_1 is real.

Finally, let us construct a regular system of real algebraic integers. We take a maximal system of real algebraic integers $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ such that $|\gamma_i - \gamma_j| > c_{14}Q^{-n}$, $1 \leq i \neq j \leq m$. Let us show that for any point $x_0 \in B_1$ there exists an algebraic number $\gamma \in \Gamma$ such that $|x_0 - \gamma| \leq 2c_{14}Q^{-n}$. From the proof presented above, for any point $x_0 \in B_1$ there exists a real algebraic integer $\alpha_1 \in I$ such that $|x_0 - \alpha_1| \leq c_{14}Q^{-n}$. If $\alpha_1 \in \Gamma$, then we can take $\gamma = \alpha_1$. If $\alpha_1 \notin \Gamma$, then there exists $\gamma_i \in \Gamma$ such that $|\alpha_1 - \gamma_i| \leq c_{14}Q^{-n}$ and

$$|x_0 - \gamma_i| \leq |x_0 - \alpha_1| + |\alpha_1 - \gamma_i| \leq 2c_{14}Q^{-n}.$$

In this case, we can take $\gamma = \gamma_i$. Therefore, B_1 is contained in a union

$$B_1 \subset \bigcup_{i=1}^m \{x \in \mathbb{R} : |x - \gamma_i| \leq 2c_{14}Q^{-n}\},$$

and

$$4c_{14}Q^{-n} \cdot m > \frac{3}{4} \cdot |I|.$$

This inequality implies that

$$m > \frac{3}{16} \cdot c_{14}^{-1} Q^n |I| = c_4 Q^n |I|$$

proving Theorem 1.2.

From the proof of Theorem 1.2 it follows, that the set of algebraic integers of degree n forms a regular system with respect to the function $N(\alpha) = H(\alpha)^n$ and $T_0 = c_{15}|I|^{-n}$, where the constant c_{15} doesn't depend on $|I|$.

4 Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the same method as the proof of Theorem 1.2, but it contains some non-trivial elements which require special attention.

The proof of Theorem 1.3 is based on Lemma 2.6. Given positive constants u_1 and u_2 satisfying the condition $u_1 + u_2 = n - 2$, let us consider a system of inequalities

$$(4.1) \quad \begin{cases} |P(x)| < Q^{-u_1}, & |P(y)| < Q^{-u_2}, \\ \min \{|P'(x)|, |P'(y)|\} < \delta_0 Q. \end{cases}$$

Lemma 2.6 implies that the measure of the set $\mathcal{M}_{n-1} = \mathcal{M}_{n-1}(Q, \delta_0, E, u_1, u_2)$ of points $(x, y) \in E$ such that the system (4.1) has a solution in polynomials $P \in \mathcal{P}_{n-1}(Q)$ can be estimated as

$$\mu_2 \mathcal{M}_{n-1} \leq \frac{1}{4} \cdot \mu_2 E$$

for $Q > Q_0(n)$ and $\delta_0 = 2^{-n-39}(n-1)^{-4}$.

Since for any point $(x, y) \in E$ there exists a polynomial $P \in \mathcal{P}_{n-1}(Q)$ satisfying $|P(x)| < Q^{-u_1}$ and $|P(y)| < Q^{-u_2}$, then for any point $(x, y) \in K_1 = E \setminus \mathcal{M}_{n-1}$ and for any polynomial $P \in \mathcal{P}_{n-1}(Q)$ the system

$$\begin{cases} |P(x)| < Q^{-u_1}, & |P(y)| < Q^{-u_1} \\ |P'(x)| \geq \delta_0 Q, & |P'(y)| \geq \delta_0 Q. \end{cases}$$

holds, and $\mu_2 K_1 \geq \frac{3}{4} \cdot \mu_2 E$.

Consider an arbitrary point $(x_0, y_0) \in K_1$ and examine the successive minima τ_1, \dots, τ_n of the compact convex set defined by

$$\begin{cases} |a_{n-1}x_0^{n-1} + \dots + a_1x_0 + a_0| \leq Q^{-u_1}, \\ |a_{n-1}y_0^{n-1} + \dots + a_1y_0 + a_0| \leq Q^{-u_2}, \\ |(n-1)a_{n-1}x_0^{n-2} + \dots + 2a_2x_0 + a_1| \leq Q, \\ |(n-1)a_{n-1}y_0^{n-2} + \dots + 2a_2y_0 + a_1| \leq Q, \\ |a_i| \leq Q, \quad 4 \leq i \leq n-1. \end{cases}$$

Assume $\tau_1 \leq \delta_0$. Then there exists a polynomial $P_0 \in \mathcal{P}_{n-1}(Q)$ such that the inequalities

$$\begin{cases} |P_0(x_0)| < \delta_0 Q^{-u_1} < Q^{-u_1}, & |P_0(y_0)| < \delta_0 Q^{-u_2} < Q^{-u_2}, \\ |P'_0(x_0)| < \delta_0 Q, & |P'_0(y_0)| < \delta_0 Q, \\ H(P_0) \leq \delta_0 Q < Q. \end{cases}$$

hold, contradicting the fact that (x_0, y_0) lies in the set K_1 . Thus, $\tau_1 > \delta_0$, allowing us to use inequality $\tau_1 \dots \tau_n \leq 1$ and Lemma 2.1 to conclude that $\tau_{n-1} \leq \delta_0^{-n+1}$. This implies that there exist n linearly independent polynomials with integer coefficients $P_i \in \mathcal{P}_{n-1}(Q)$, $1 \leq i \leq n$, satisfying the inequalities

$$(4.2) \quad \begin{cases} |P_i(x_0)| \leq \delta_0^{-n+1} Q^{-u_1}, & |P_i(y_0)| \leq \delta_0^{-n+1} Q^{-u_2} \\ |P'_i(x_0)| \leq \delta_0^{-n+1} Q, & |P'_i(y_0)| \leq \delta_0^{-n+1} Q, \\ H(P_i) \leq \delta_0^{-n+1} Q. \end{cases}$$

Well-known bounds from geometry of numbers (see [15, pp. 219]) yield the following inequality for the polynomials $P_i(t) = a_{i,n-1}t^{n-1} + \dots + a_{i,1}t + a_{i,0}$, $1 \leq i \leq n$:

$$\Delta = \det |(a_{i,j-1})_{i,j=1}^n| \leq n!,$$

From Lemma 2.2, there exists a prime p which doesn't divide Δ such that

$$(4.3) \quad n! < p < 2n!.$$

Let us consider a system of linear equations in n variables $\theta_1, \dots, \theta_n$

$$(4.4) \quad \begin{cases} x_0^n + p \cdot \sum_{i=1}^n \theta_i P_i(x_0) = p(n+1) \cdot \delta_0^{-n+1} Q^{-u_1}, \\ y_0^n + p \cdot \sum_{i=1}^n \theta_i P_i(y_0) = p(n+1) \cdot \delta_0^{-n+1} Q^{-u_2}, \\ nx_0^{n-1} + p \cdot \sum_{i=1}^n \theta_i P'_i(x_0) = pQ + p \cdot \sum_{i=1}^n |P'_i(x_0)|, \\ ny_0^{n-1} + p \cdot \sum_{i=1}^n \theta_i P'_i(y_0) = pQ + p \cdot \sum_{i=1}^n |P'_i(y_0)|, \\ \sum_{i=1}^n \theta_i a_{i,j} = 0, \quad 4 \leq j \leq n-1. \end{cases}$$

In order to find the determinant of this system, let us transform it as follows. Multiply the equations numbered as $k = 5, 6, \dots, n$ by $p \cdot x_0^{k-1}$ (respectively by $p \cdot y_0^{k-1}$) and subtract them from the first (respectively the second) equation of the system (4.4). Similarly, multiply the equations numbered as $k = 5, 6, \dots, n$ by $p \cdot (k-1)x_0^{k-2}$ (respectively by $p \cdot (k-1)y_0^{k-2}$) and subtract them from the third (respectively the fourth) equation. After

these transformations the determinant of system (4.4) can be written as

$$\hat{\Delta} = p^4 \cdot \begin{vmatrix} \sum_{i=0}^3 a_{1,i} x_0^i & \dots & \sum_{i=0}^3 a_{n,i} x_0^i \\ \sum_{i=0}^3 a_{1,i} y_0^i & \dots & \sum_{i=0}^3 a_{n,i} y_0^i \\ \sum_{i=1}^3 i \cdot a_{1,i} x_0^{i-1} & \dots & \sum_{i=1}^3 i \cdot a_{n,i} x_0^{i-1} \\ \sum_{i=1}^3 i \cdot a_{1,i} y_0^{i-1} & \dots & \sum_{i=1}^3 i \cdot a_{n,i} y_0^{i-1} \\ a_{1,4} & \dots & a_{n,4} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix}$$

Let us transform the first four rows of this matrix as follows. Multiply the third (respectively the fourth) row by $\frac{1}{3}x_0$ (respectively by $\frac{1}{3}y_0$) and subtract it from the first (respectively the second) row. Then subtract the first (respectively the third) row from the second (respectively the fourth) row we obtain the following determinant:

$$\frac{p^4(y_0 - x_0)^2}{9} \cdot \begin{vmatrix} a_{1,2}x_0^2 + 2a_{1,1}x_0 + 3a_{1,0} & \dots & a_{n,2}x_0^2 + 2a_{n,1}x_0 + 3a_{n,0} \\ a_{1,2}(y_0 + x_0) + 2a_{1,1} & \dots & a_{n,2}(y_0 + x_0) + 2a_{n,1} \\ 3a_{1,3}x_0^2 + 2a_{1,2}x_0 + a_{1,1} & \dots & 3a_{n,3}x_0^2 + 2a_{n,2}x_0 + a_{n,1} \\ 3a_{1,3}(y_0 + x_0) + 2a_{1,2} & \dots & 3a_{n,3}(y_0 + x_0) + 2a_{n,2} \\ a_{1,4} & \dots & a_{n,4} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix}.$$

Now let us subtract the second row multiplied by x_0 from the first row. Similarly, subtract the fourth row multiplied by $\frac{1}{2}$ from the third row. Then subtract the third row multiplied by $\frac{y_0 + x_0}{x_0^2}$ from the fourth row, and finally subtract the fourth row multiplied by $x_0 y_0$, $y_0 + x_0$ and $\frac{3}{2}x_0 - \frac{1}{2}y_0$ from the first, the second and the third row respectively. We obtain the equation

$$(4.5) \quad \hat{\Delta} = p^4(y_0 - x_0)^4 \cdot \begin{vmatrix} a_{1,0} & \dots & a_{n,0} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix} = p^4(y_0 - x_0)^4 \Delta > 0,$$

since the polynomials P_i , $1 \leq i \leq n$ are linearly independent, and $|y_0 - x_0| > \varepsilon > 0$. By (4.5), the system (4.4) has a unique solution $(\theta_1, \dots, \theta_n)$. Consider n integers t_1, \dots, t_n such that

$$(4.6) \quad |\theta_i - t_i| \leq 1, \quad i = 1, \dots, n,$$

as well as the following polynomial with integer coefficients:

$$P(t) = t^n + p \cdot \sum_{i=1}^n t_i P_i(t) = t^n + p \cdot (a_{n-1}t^{n-1} + \dots + a_1t + a_0),$$

where $a_j = \sum_{i=1}^n t_i a_{i,j}$, $0 \leq j \leq n-1$. By Lemma 2.3 and the arguments from the previous section, we can choose the values t_j so that the polynomial P is irreducible.

Finally, let us obtain bounds for $|P(x_0)|$, $|P(y_0)|$, $|P'(x_0)|$ and $|P'(y_0)|$. By inequalities (4.2), (4.4) and (4.6), we have the following estimates:

$$(4.7) \quad p\delta_0^{-n+1}Q^{-u_1} \leq |P(x_0)| \leq p(2n+1)\delta_0^{-n+1}Q^{-u_1},$$

$$(4.8) \quad p\delta_0^{-n+1}Q^{-u_2} \leq |P(y_0)| \leq p(2n+1)\delta_0^{-n+1}Q^{-u_2},$$

$$(4.9) \quad pQ \leq |P'(x_0)| \leq (p + 2pn\delta_0^{-n+1})Q,$$

$$(4.10) \quad pQ \leq |P'(y_0)| \leq (p + 2pn\delta_0^{-n+1})Q.$$

Let us estimate the height $H(P)$. By equations 4 to n of the system (4.4) and inequalities (4.2), (4.6), we have

$$(4.11) \quad |a_j| \leq n\delta_0^{-n+1}Q, \quad 4 \leq j \leq n-1.$$

It remains to estimate $|a_j|$, $0 \leq j \leq 3$. By (4.7) – (4.11) and the inequalities $|x_0| \leq \frac{1}{2}$, $|y_0| \leq \frac{1}{2}$, we have

$$(4.12) \quad \begin{cases} |a_3x_0^3 + a_2x_0^2 + a_1x_0 + a_0| \leq |P(x_0)| + \sum_{j=4}^n |a_j| < 2pn\delta_0^{-n+1}Q, \\ |a_3y_0^3 + a_2y_0^2 + a_1y_0 + a_0| \leq |P(y_0)| + \sum_{j=4}^n |a_j| < 2pn\delta_0^{-n+1}Q, \\ |3a_3x_0^2 + 2a_2x_0 + a_1| \leq |P'(x_0)| + \sum_{j=4}^n j \cdot |a_j| < 2pn^3\delta_0^{-n+1}Q, \\ |3a_3y_0^2 + 2a_2y_0 + a_1| \leq |P'(y_0)| + \sum_{j=4}^n j \cdot |a_j| < 2pn^3\delta_0^{-n+1}Q. \end{cases}$$

Consider the following system of linear equations for a_0 , a_1 , a_2 and a_3 :

$$(4.13) \quad \begin{cases} a_3x_0^3 + a_2x_0^2 + a_1x_0 + a_0 = l_1, \\ a_3y_0^3 + a_2y_0^2 + a_1y_0 + a_0 = l_2, \\ 3a_3x_0^2 + 2a_2x_0 + a_1 = l_3, \\ 3a_3y_0^2 + 2a_2y_0 + a_1 = l_4. \end{cases}$$

Since the determinant of the system (4.13) does not vanish, the system has a unique solution. Let us solve the system (4.13) using the estimates (4.12) and the inequalities $|x_0| \leq \frac{1}{2}$, $|y_0| \leq \frac{1}{2}$. We obtain

$$|a_j| < 10^4pn^3\delta_0^{-n+1}Q, \quad 0 \leq j \leq 3.$$

Applying by (4.3) and (4.11) yields the following estimate for the height of P :

$$H(P) < 2 \cdot 10^4 (n+4)! \delta_0^{-n+1} Q.$$

Consider roots $\alpha_1, \dots, \alpha_n$ of the polynomial P , where $|x_0 - \alpha_1| = \min_i |x_0 - \alpha_i|$ and let β_1, \dots, β_n be a permutation of these roots such that $|y_0 - \beta_1| = \min_i |y_0 - \beta_i|$. By Lemma 2.4, the following estimates hold:

$$\begin{cases} |x_0 - \alpha_1| \leq n |P(x_0)| |P'(x_0)|^{-1}, \\ |y_0 - \beta_1| \leq n |P(y_0)| |P'(y_0)|^{-1}. \end{cases}$$

By (4.7) – (4.9), we have

$$\begin{cases} |x_0 - \alpha_1| < n(2n+1) \delta_0^{-n+1} Q^{-u_1-1} < c_{16} Q^{-u_1-1}, \\ |y_0 - \beta_1| < n(2n+1) \delta_0^{-n+1} Q^{-u_2-1} < c_{16} Q^{-u_2-1}, \end{cases}$$

where $c_{16} = n(2n+1) \delta_0^{-n+1}$. For $Q > Q_0(n)$, the roots α_1 and β_1 are real.

Let $\Gamma = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ be a maximal set of real algebraic integer points such that

$$|\alpha_i - \alpha_j| > c_{16} Q^{-u_1-1} \quad \text{or} \quad |\beta_i - \beta_j| > c_{16} Q^{-u_2-1}, \quad 1 \leq i \neq j \leq m.$$

Using the arguments presented in the previous section, we can prove that for any point $(x_0, y_0) \in K_1$ there exists an algebraic integer point $(\alpha_i, \beta_i) \in \Gamma$ such that

$$\begin{cases} |x_0 - \alpha_i| < 2c_{16} Q^{-u_1-1}, \\ |y_0 - \beta_i| < 2c_{16} Q^{-u_2-1}. \end{cases}$$

This implies the following covering:

$$K_1 \subset \bigcup_{i=1}^m \{(x, y) \in \mathbb{R}^2 : |x - \alpha_i| < 2c_{16} Q^{-u_1-1}, |y - \beta_i| < 2c_{16} Q^{-u_2-1}\},$$

where

$$m > \frac{3}{64} \cdot c_{16}^{-2} Q^n \mu_2 E = c_7 Q^n \mu_2 E.$$

Theorem 1.3 has been proved.

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